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Comment on "Minimal atmospheric finite-mode models preserving symmetry and generalized Hamiltonian structures, Physica D 240 (2011) 599–606"

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ABSTRACT

Bihlo and Staufer (2011), point out that "truncation to systems in coupled gyrostats form (Gluhovsky, 2006, Gluhovsky et al., 2002) may also lead to models that retain the conservation properties of the original equations and that a single gyrostat is a Nambu system and hence using such a truncation, conservation of the underlying geometry may be implemented at least in some minimal form". In this note, we demonstrate that example systems in Bihlo and Staufer (2011) may indeed be treated in terms of gyrostats; in particular, their central example (six-mode system (13)) proves to be a four-dimensional gyrostat.

1. Introduction

In a very interesting paper, Bihlo and Staufer [1] considered a fundamental problem of developing finite-mode models that retain essential properties of the original hydrodynamic equations. They argued, in particular, that while the three-mode Lorenz-1960 model [2] of the barotropic vorticity equation retains point symmetries and the associated Nambu form of the latter, the celebrated three-mode Lorenz-1963 model [3] of 2D Rayleigh–Bénard convection is deficient in this regard, and suggested instead its six-mode extension with the conservative core (Eqs. (13) in [1]),

$$
\begin{aligned}
\frac{dA}{dt} &= \frac{a}{2b\pi(1+a)^2}((a^2-3)\pi^3 BC + 2eR\sigma E),\\
\frac{dB}{dt} &= -\frac{a}{2b\pi(1+a)^2}((a^2-3)\pi^3 AC + 2eR\sigma D),\\
\frac{dC}{dt} &= 0,\\
\frac{dD}{dt} &= \frac{a\pi}{2be}(e\pi CE - 2b^2 f\pi BF - 2b^2 B),\\
\frac{dE}{dt} &= -\frac{a\pi}{2be}(e\pi CD - 2b^2 f\pi AF - 2b^2 A),\\
\frac{dF}{dt} &= \frac{abe\pi^2}{2f}(BD - AE).
\end{aligned}
\tag{1}
$$

It turns out that all these systems may be treated in terms of gyrostats.

1.1. Volterra gyrostat, Lorenz-1960 and Lorenz-1963 models

The Volterra gyrostat [4,5]

$$
\begin{aligned}
I_1\dot{\omega}_1 &= (I_2-I_3)\omega_2\omega_3 + h_2\omega_3 - h_3\omega_2,\\
I_2\dot{\omega}_2 &= (I_3-I_1)\omega_3\omega_1 + h_3\omega_1 - h_1\omega_3,\\
I_3\dot{\omega}_3 &= (I_1-I_2)\omega_1\omega_2 + h_1\omega_2 - h_2\omega_1,
\end{aligned}
\tag{2}
$$

where the overdot means differentiation with respect to time, is a classic mechanical system, which, in one of its simplest forms, was shown [6] to be equivalent to the Lorenz-1963 model, whereas systems of coupled gyrostats found various applications in geophysical fluid dynamics including shell models of turbulence and Hamiltonian low-order models [6–14]. System (2) can be thought of as a rigid body containing an axisymmetric rotor that rotates with a constant angular velocity about an axis fixed in the carrier. In this interpretation, I_i, $i = 1, 2, 3$, in Eqs. (2) are the principal moments of inertia of the gyrostat, $\boldsymbol{\omega}$ is the angular velocity of the carrier body, and $\boldsymbol{h}$ is the fixed angular momentum caused by the relative motion of the rotor.

System (2) without linear terms ($h_1 = h_2 = h_3 = 0$) is simply the Euler rigid body, equivalent to the Lorenz-1960 model and to the Obukhov triplet [15,16].

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Of particular interest is the simplest gyrostat (system (2) with only two nonlinear and two linear terms),

$$\begin{aligned} I_1\dot{\omega}_1 &= -(I_3-I_1)\omega_2\omega_3, \\ I_2\dot{\omega}_2 &= \quad (I_3-I_1)\omega_3\omega_1 - h_1\omega_3, \\ I_3\dot{\omega}_3 &= \qquad\qquad\qquad\quad\; h_1\omega_2, \end{aligned} \tag{3}$$

since in the forced regime (i.e., with added forcing and friction) it is equivalent to the Lorenz-1963 model [6] (see, also, [9]). Eqs. (3) are obtained by setting

$$I_1 = I_2, \qquad h_2 = h_3 = 0, \tag{4}$$

in Eqs. (2), which means, in the above interpretation of the Volterra gyrostat as a system of two bodies, that the rotor spins about one of the principal axes of the system, while the ellipsoid of inertia is the ellipsoid of rotation about another principal axis.

1.2. n-dimensional gyrostat

Similar to the Arnold definition [17] of the n-dimensional rigid body as a quadratically nonlinear system on the Lie algebra of the group $SO(n)$, the n-dimensional gyrostat was introduced [6] as the n-dimensional analog of the Volterra equations (2),

$$(\lambda_i+\lambda_j)\dot{\Omega}_{ij} = \sum_{k\neq i,j}[(\lambda_i+\lambda_j)\Omega_{ik}\Omega_{kj} + h_{ik}\Omega_{kj} - h_{kj}\Omega_{ik}],$$

$$i\neq j,\ 1\leqslant i,j,k\leqslant n,\ \lambda_i>0,\ \Omega_{ij}=-\Omega_{ji},\ h_{ij}=-h_{ji}. \tag{5}$$

For example, Eqs. (2) result from Eqs. (5) by setting $n=3$ and

$$\begin{aligned} &I_1=\lambda_2+\lambda_1, \qquad I_2=\lambda_1+\lambda_3, \qquad I_3=\lambda_3+\lambda_2; \\ &\omega_4=\Omega_{41}, \qquad \omega_5=\Omega_{42}, \qquad \omega_6=\Omega_{43}; \\ &h_4=h_{41}, \qquad h_5=h_{42}, \qquad h_6=h_{43}, \end{aligned} \tag{6}$$

while at $n=4$ and

$$\begin{aligned} &I_4=\lambda_4+\lambda_1, \qquad I_5=\lambda_4+\lambda_2, \qquad I_6=\lambda_4+\lambda_3; \\ &\omega_4=\Omega_{41}, \qquad \omega_5=\Omega_{42}, \qquad \omega_6=\Omega_{43}; \\ &h_4=h_{41}, \qquad h_5=h_{42}, \qquad h_6=h_{43} \end{aligned} \tag{7}$$

in addition to (6), the equations for the four-dimensional gyrostat take the form

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2-I_3)\omega_2\omega_3 + (I_2-I_3)\omega_4\omega_5 + h_2\omega_3 \\ &\quad - h_3\omega_2 + h_4\omega_5 - h_5\omega_4, \\ I_2\dot{\omega}_2 &= (I_3-I_1)\omega_3\omega_1 + (I_4-I_6)\omega_4\omega_6 + h_3\omega_1 \\ &\quad - h_1\omega_3 + h_4\omega_6 - h_6\omega_4, \\ I_3\dot{\omega}_3 &= (I_1-I_2)\omega_1\omega_2 + (I_5-I_3)\omega_5\omega_3 + h_1\omega_2 \\ &\quad - h_2\omega_1 + h_5\omega_6 - h_6\omega_5, \\ I_4\dot{\omega}_1 &= (I_5-I_1)\omega_5\omega_1 + (I_6-I_2)\omega_6\omega_2 + h_5\omega_1 \\ &\quad - h_1\omega_5 + h_6\omega_2 - h_2\omega_6, \\ I_5\dot{\omega}_2 &= (I_1-I_4)\omega_1\omega_4 + (I_6-I_3)\omega_6\omega_3 + h_1\omega_4 \\ &\quad - h_4\omega_1 + h_6\omega_3 - h_3\omega_6, \\ I_6\dot{\omega}_3 &= (I_2-I_4)\omega_4\omega_2 + (I_3-I_5)\omega_3\omega_5 + h_4\omega_2 \\ &\quad - h_2\omega_4 + h_3\omega_5 - h_5\omega_3. \end{aligned} \tag{8}$$

2. Four-dimensional gyrostat with symmetries and system (1)

Eqs. (3) were obtained from Eqs. (2) by imposing conditions (4), i.e., by setting $\lambda_2=\lambda_3$ and $h_2=h_3=0$ in Eqs. (5) at $n=3$. Similarly, to obtain a gyrostatic form of system (1) from Eqs. (8), let $n=4$ and

$$I_4=I_2, \qquad I_5=I_6=I_3, \qquad h_4=h_5=0, \tag{9}$$

in addition to (6) and (4), i.e.,

$$\lambda_2=\lambda_3=\lambda_4; \qquad h_2=h_3=h_4=h_5=0 \tag{10}$$

in Eqs. (5). Then Eqs. (8) become,

$$\begin{aligned} I_1\dot{\omega}_1 &= -(I_3-I_1)\omega_2\omega_3 - (I_3-I_1)\omega_2\omega_3, \\ I_1\dot{\omega}_2 &= (I_3-I_1)\omega_3\omega_1 - h_1\omega_3 - \tilde{h}_6\omega_4, \\ I_3\dot{\omega}_3 &= h_1\omega_2 - h_6\omega_5, \\ I_1\dot{\omega}_4 &= (I_3-I_1)\omega_5\omega_1 - h_1\omega_5 + \tilde{h}_6\omega_2, \\ I_3\dot{\omega}_5 &= h_1\omega_6 + h_6\omega_3, \\ I_3\dot{\omega}_6 &= 0, \end{aligned} \tag{11}$$

where $\tilde{h}_6=(I_3-I_1)\omega_6+h_6$, and by the change of variables,

$$\begin{aligned} &\omega_1=x_1\sqrt{I_1}, \qquad \omega_2=x_2\sqrt{I_1}, \qquad \omega_3=x_3\sqrt{I_3}, \\ &\omega_4=x_4\sqrt{I_1}, \qquad \omega_5=x_5\sqrt{I_3}, \qquad \omega_6=x_6\sqrt{I_3}, \end{aligned} \tag{12}$$

and time, $\tau = I_1\sqrt{I_3}/(I_3-I_1)$, the resulting four-dimensional gyrostat takes the following form,

$$\begin{array}{l|l|l|l|l|} \dot{x}_1= & -x_2x_3 & -x_4x_5 & & \\ \dot{x}_2= & x_3x_1-c_1x_3 & & -c_2x_4 & \\ \dot{x}_3= & \quad c_1x_2 & & & -c_3x_5 \\ \dot{x}_4= & & x_5x_1-c_1x_5 & +c_2x_2 & \\ \dot{x}_5= & & \quad c_1x_4 & & +c_3x_3 \\ \dot{x}_6=0 & & & & \end{array} \tag{13}$$

where $c_1=h_1\sqrt{I_1}/(I_3-I_1)$, $c_2=x_6+h_6\sqrt{I_3}/(I_3-I_1)$, $c_3=h_6I_1/\sqrt{I_3}(I_3-I_1)$. The vertical bars in Eqs. (13) separate four subsystems: two gyrostats (3) and two linear oscillators. Interestingly, the system composed of the two gyrostats (i.e., the first five equations (13) without linear terms) becomes in the forced regime the simplest model of 3D Rayleigh–Bénard convection [10], where two Lorenz-1963 models describe the dynamics in the (x,z) and (y,z) planes, respectively.

Finally, in the variables

$$\begin{aligned} &x_1=p(F+\alpha), \qquad x_2=qA, \qquad x_3=rE, \qquad x_4=-qB, \\ &x_5=rD, \qquad x_6=C, \end{aligned} \tag{14}$$

where α is an arbitrary constant subject to the condition $\beta = \pi f\alpha-1>0$, and $p=\pi af\sqrt{R\sigma/\beta(1+a^2)}$, $q=\pi^2ab/\sqrt{2}$, $r=(e/\sqrt{2f})p$, system (1) also turns into the four-dimensional gyrostat (13) with

$$\begin{aligned} &c_1=a\sqrt{\frac{R\sigma\beta}{1+a^2}}, \qquad c_2=\frac{\pi}{be}\sqrt{\frac{\beta(1+a^2)}{2R\sigma}}C, \\ &c_3=\frac{a^2-3}{\sqrt{2}b^2(1+a^2)}C. \end{aligned} \tag{15}$$

3. Concluding remarks

We have shown that the central example in [1] is a four-dimensional gyrostat. Since various energy-conserving and Hamiltonian low-order models in geophysical fluid dynamics have turned out to be *gyrostatic systems* (Volterra gyrostats, systems of coupled Volterra gyrostats, n-dimensional gyrostats) [6–14], such systems may prove useful for exploring physically sound Galerkin approximations in general. Additionally, for Fourier series expansion different from that employed by Bihlo and Staufer (Eqs. (11) in [1]), Treve and Manley [18] suggested a mode selection procedure always resulting in energy conserving models, all having the form of coupled gyrostats [7,19].

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